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Invariance Analysis and Variational Conservation Laws for the Wave Equation on Some Manifolds

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Abstract In this paper we discuss symmetries of a nonlinear wave equation that arises as a consequence of some Riemannian metrics of signature -2. The objective of this study is to show how geometry can be responsible in giving rise to a nonlinear inhomogeneous wave equation rather than assuming nonlinearities in the wave equation from physical considerations. We find Lie point symmetries of the corresponding wave equations and give their solutions in two cases. Some interesting physical conclusions relating to conservation laws such as energy, linear and angular momenta are also determined.

Keywords Symmetries · Conservation laws

1 Introduction

The wave equation has been extensively studied in literature from the point of view of its Lie point symmetries. A detailed symmetry analysis of this equation is discussed by Cantwell [1], Ibragimov [2] and Bluman and Kumei [3]. It is well known that in three Euclidean space dimensions the linear wave equation admits maximal *'16-dimensional'* Lie algebra of point symmetries excluding the 'infinite symmetry' [4]. Realistically, one would expect that a genuinely interesting wave equation will possess non-linearities. Whereas the non-linearities allow the wave equation to represent physically interesting situations, the difficult

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part is to justify introduction of nonlinearities? Generally nonlinearities in the wave equation are introduced by keeping in mind physical considerations such as properties of media etc. In this work we detour from this approach and using a purely geometric consideration construct the wave equation in a curved back ground geometry in such a way that the wave equation inherits nonlinearity of the respective geometry in a natural way. Keeping in mind that wave equation in 4-spacetime dimensions may be of more physical significance, we use Lorentzian geometries of signature -2 to construct the wave equation there. In the hope that such a study may yield interesting results, we begin by focussing our attention on two specific Lorentzian geometries of which one admits plane symmetric static symmetry and the other spherically symmetric non-static symmetry [5–7]. (The symmetry of a Lorentzian metric g_{ij} is defined through Killing vectors and a Killing vector '**x**' is the one relative to which the Lie derivative of the metric g_{ij} is zero, i.e., $L_X g_{ij} = 0$ [5, 6, 8].) Generally, the wave equation constructed in a genuinely curved spacetime background will also inherit nonlinearities of that spacetime. One method to solve this equation is to first linearize the equation and then solve it. However, such an approach will generally dilute the elegance of nonlinearity of that geometry. To solve the equation with geometric nonlinearity intact, we use Lie point symmetry method [1, 3, 10]. This method allows to solve the nonlinear wave equations with out linearizing it and is based upon finding the Lie symmetries of the equation, using these symmetries to reduce the wave equation to an ordinary differential equation and solving the resulting differential equation. Since we are dealing with wave equations in curved back ground settings, it may also be interesting to look into some conservation laws admitted by the spacetime metrics. With this point in mind, we conclude work by constructing certain conservation laws for the two spacetime geometries via the Noether theorem [12]. The plan of the paper is as follows:

In the next section we derive the wave equation in a plane symmetric static spacetime metric, find its Lie point symmetries, derive some exact solutions of the equation and construct Noether symmetries that arise from a 'natural' Lagrangian. In Sect. 3 a similar analysis is presented of the wave equation in spherically symmetric non-static flat Friedman metric [8]. In Sect. 4 a procedure to construct variational conservation laws is given and some illustrations given. A brief summary and discussion is given in the last section.

2 Wave Equation in a Plane Symmetric Static Spacetime Background

A wave equation on a Lorentzian manifold endowed with a metric g_{ij} is given by the expression [9],

$$\Box u(\underline{x}, t) = g^{00}\partial_0^2 + \frac{1}{2}g^{ij}[g^{00}(\partial_i g_{00})\partial_j + \partial_i\partial_j - \Gamma^k_{ij}\partial_k]u(\underline{x}, t) = 0, \tag{2.1}$$

where $u(\underline{x}, t)$ is some given wave function, $\Gamma^k_{ij} = \frac{1}{2}g^{km}(\partial_j g_{im} + \partial_i g_{jm} - \partial_m g_{ij})$ represents Christoffel symbol, g^{ij} inverse of the metric g_{ij} and $\underline{x}$ three space variables x, y, z. In this section we chose a particular form of plane symmetric static spacetime. This is the Minkowski metric in some new coordinates and was found in the process of classification of such spacetimes by Kashif in [7]

$$g_{ij} = ((x/a)^2, -1, -1, -1). \tag{2.2}$$

Using above metric the wave equation (2.1) takes the form,

$$u_{tt} = \frac{x^2}{a^2}(u_{xx} + u_{yy} + u_{zz}) + \frac{x}{a^2}u_x \tag{2.3}$$

To solve (2.3) we use Lie point symmetry method [3]. This method rests on finding Lie point symmetry transformations which leave the wave equation (2.1) invariant. These transformations are given by [1, 3, 10]

$$\widetilde{\mathbf{x}} = \mathbf{x} + \epsilon \xi_i(\mathbf{x}, u) + O(\epsilon^2) \quad \text{for } i = 0, \dots, 3 \tag{2.4}$$

where $\mathbf{x}$ and ξ_i respectively represent variables on which the wave equation depends and corresponding components of the symmetry generator associated with the basis x, y, z, t, u. Similarly, using (2.4), the expressions for the derivatives of the transformed 'dependent' variables with respect to the transformed 'independent' variables become:

$$\widetilde{u}_j = u_j + \epsilon \phi^j(\mathbf{x}, u) + O(\epsilon^2) \quad \text{for } j = 1, 2, \dots \tag{2.5}$$

where $u_j = \frac{\partial u}{\partial x^j}$. To solve (2.3), we start by writing symmetry generator corresponding to the variables x, y, z, t and u

$$V = m\frac{\partial}{\partial x} + n\frac{\partial}{\partial y} + p\frac{\partial}{\partial z} + q\frac{\partial}{\partial t} + s\frac{\partial}{\partial u} \tag{2.6}$$

where m, n, p and q and s are components of the symmetry generator V computed for $i = 0, \dots, 3$ at $\epsilon = 0$. With the symmetry generator (2.6) in hand, the next step is to prolong it to second order [1, 3, 10]. In the four spacetime coordinates the prolonged symmetry generator becomes

$$V^{(1)} = V + s_i \frac{\partial}{\partial u_i} + s_{ij}\frac{\partial}{\partial u_{ij}}, \tag{2.7}$$

where coefficient 'of the prolonged generator' is a function of (x, y, z, t, u) and are determined by the formulae,

$$\begin{aligned} s_i &= D_i(\phi - \xi^j u_{ji}) + \xi^j u_{j,i}, \\ s_{ij} &= D_i D_j(\phi - \xi^j u_{ji}) + \xi^k u_{k,ij}, \end{aligned} \tag{2.8}$$

where D_i represents total derivative and subscripts of u derivative with respect to the respective coordinates and $i = j = x, y, z, t$. At this stage we use symmetry criterion $V^1[u_{tt} - \frac{x^2}{a^2}(u_{xx} + u_{yy} + u_{zz}) - \frac{x}{a^2}u_x] |_{eq.(2.3)=0}$ for the wave equation. Using this condition and (2.8), replacing u_{tt} into the resulting expression and then comparing coefficients of all derivatives and products of derivative of u, gives rise to an over-determined system of partial differential equations:

$$\begin{aligned}
&m_u = 0 = n_u = p_u = q_u = q_{u,u} = s_{u,u} \\
&a^2 m_t - x^2 q_x = 0, \qquad a^2 n_t - x^2 q_y = 0, \qquad a^2 p_t - x^2 q_z = 0 \\
&m - x m_x + x q_t = 0, \qquad m - x n_y + x q_t = 0, \qquad m - x p_z + x q_t = 0 \\
&m_y + n_x = 0, \qquad m_z + p_x = 0, \qquad n_z + p_y = 0 \\
&m + 2x q_t - x m_x + a^2 m_{t,t} - x^2(m_{x,x} + m_{y,y} + m_{z,z} - 2 s_{x,u}) = 0 \\
&x n_x - a^2 n_{t,t} + x^2(n_{x,x} + n_{y,y} + n_{z,z} - 2 s_{y,u}) = 0 \\
&x p_x - a^2 p_{t,t} + x^2(p_{x,x} + p_{y,y} + p_{z,z} - 2 s_{z,u}) = 0 \\
&x q_x - a^2 q_{t,t} + x^2(q_{x,x} + q_{y,y} + q_{z,z} - 2 s_{t,u}) = 0 \\
&x s_x - a^2 s_{t,t} + x^2(s_{x,x} + s_{y,y} + s_{z,z}) = 0
\end{aligned} \tag{2.9}$$

One way to solve the above system is to follow the ab-initio method [10]. However, keeping in mind that this is a routine calculation, we use an algebraic software [11] to solve this system. Using this software suggests that the above system has 15 independent solutions (also called symmetries admitted by (2.3)) and give rise to a Lie algebra of 15 Lie point symmetries. Also the equation admits an additional arbitrary infinite-dimensional symmetry generator, $\phi(t,x,y,z)\partial_u$ in ϕ which satisfies the wave equation (2.3). The 15 Lie point symmetry generators are given in the following equations:

$$\begin{aligned}
&X_0 = \frac{\partial}{\partial t}, \qquad X_1 = \frac{\partial}{\partial y}, \qquad X_2 = \frac{\partial}{\partial z}\\
&X_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} - u\frac{\partial}{\partial u}, \qquad X_4 = u\frac{\partial}{\partial u}\\
&X_5 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \qquad X_6 = e^{\frac{t}{a}}\left(\frac{\partial}{\partial x} - \frac{a}{x}\frac{\partial}{\partial t}\right), \qquad X_7 = e^{-\frac{t}{a}}\left(\frac{\partial}{\partial x} + \frac{a}{x}\frac{\partial}{\partial t}\right)\\
&X_8 = e^{\frac{t}{a}}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} - a\frac{z}{x}\frac{\partial}{\partial t}\right), \qquad X_9 = e^{-\frac{t}{a}}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} + a\frac{z}{x}\frac{\partial}{\partial t}\right)\\
&X_{10} = e^{\frac{t}{a}}\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} - a\frac{y}{x}\frac{\partial}{\partial t}\right), \qquad X_{11} = e^{-\frac{t}{a}}\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + a\frac{y}{x}\frac{\partial}{\partial t}\right)\\
&X_{12} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + u\frac{\partial}{\partial u}\\
&X_{13} = 2xy\frac{\partial}{\partial x} + (y^2 - x^2 - z^2)\frac{\partial}{\partial y} + 2yz\frac{\partial}{\partial z} - 2uy\frac{\partial}{\partial u}\\
&X_{14} = 2xz\frac{\partial}{\partial x} + 2yz\frac{\partial}{\partial y} + (z^2 - x^2 - y^2)\frac{\partial}{\partial z} - 2uz\frac{\partial}{\partial u}
\end{aligned} \tag{2.10}$$

At this stage we use a subalgebra of the 15 generators to reduce (2.3) to an ordinary differential equation. Since we are dealing with wave equation in 4 spacetime variables, a three-dimensional subalgebra of (2.10) is required to reduce (2.3) to an ordinary differential equation. With this point in mind we choose three symmetry generators $\{X_5, X_6, X_{12}\}$ which satisfy the commutation relation given by,

$$[X_5, X_6] = 0, \qquad [X_6, X_{12}] = X_6, \qquad [X_5, X_{12}] = 0.$$

Since our purpose is to show how solutions of the wave equation in a Lorentzian geometry can be found using the Lie symmetry method, we present solution in only one case. All other solutions can be found in a similar fashion. To find one solution we begin with the subalgebra given by $[X_5, X_6] = 0$. Since symmetry generators commute, we can use either of the two generators to proceed with finding solution of the wave equation [1]. We start with X_5 and write its characteristic equation as,

$$\frac{dx}{0} = \frac{dy}{z} = \frac{dz}{-y} = \frac{dt}{0} = \frac{du}{0} \tag{2.11}$$

Considering $\frac{dy}{z} = \frac{dz}{-y}$ from (2.11) and integrating gives $\alpha = y^2 + z^2$. The remaining part of characteristic equation (2.11) suggests that $u = u(\alpha)$. Now we re-write wave equation (2.3) in terms of the new variable α. This can be done by expressing derivatives of the dependent variable 'u' in terms of each of y and z given by $u_y = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial y} = 2yu_\alpha$ and $u_z = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial z} = 2zu_\alpha$. Following this procedure one finds that $u_{yy} = 2u_\alpha + 4y^2 u_{\alpha\alpha}$ and $u_{zz} = 2u_\alpha + 4z^2 u_{\alpha\alpha}$ so that $\alpha = 4u_\alpha + 4(y^2+z^2)u_{\alpha\alpha}$. In the light of these (2.3) becomes,

$$u_{tt} = \frac{x^2}{a^2}(u_{xx} + 4\alpha u_{\alpha\alpha} + 4u_\alpha) + \frac{x}{a^2}u_x. \tag{2.12}$$

We now use symmetry generator X_6 for further reduction. The characteristic equation for this generator is,

$$\frac{dx}{e^{t/a}} = \frac{dt}{-e^{t/a}(a/x)} = \frac{d\alpha}{0} \tag{2.13}$$

Solving above characteristic equation similar to (2.13) yields $\beta = t + a\ln x$ with $u = u(\beta)$. From here we express u_t, u_x, u_{tt} and u_{xx} in terms of new variables given respectively by, $u_t = u_\beta$, $u_x = (a/x)u_\beta$, $u_{tt} = u_{\beta\beta}$ and $u_{xx} = (a^2/x^2)u_{\beta\beta} - (a/x^2)u_\beta$. These transformations reduce (2.12) to

$$\alpha u_{\alpha\alpha} + u_\alpha = 0. \tag{2.14}$$

With expressions for α and β found, we are now left with using $X_{12} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + u\frac{\partial}{\partial u}$. To use it for solving the wave equation we re-write it in new variables α and β. In these variables four terms $x\frac{\partial}{\partial x}$, $y\frac{\partial}{\partial y}$, $z\frac{\partial}{\partial z}$ and $u\frac{\partial}{\partial u}$ comprising X_{12} respectively become $a\frac{\partial}{\partial \beta}$, $2y^2\frac{\partial}{\partial \alpha}$ and $2z^2\frac{\partial}{\partial \alpha}$ with $u\frac{\partial}{\partial u}$ remaining unchanged. Using these into X_{12} transform it to $\hat{X}_{12} = a\partial_\beta + 2\alpha\partial_\alpha + u\partial_u$. Solving its characteristic equation gives rise to the invariants $\gamma = \ln\alpha - \frac{2}{a}\beta$ and u. Using this into equation (2.14) reduces it to a second order ordinary differential equation

$$u_{\gamma\gamma} = 0, \tag{2.15}$$

whose solution is $u = A\gamma + B$ where A and B are arbitrary constants of integration. Lastly, we re-cast this solution in original coordinates as

$$u = A\left[\ln(y^2+z^2) - \frac{2}{a}(t + a\ln x)\right] + B. \tag{2.16}$$

This is an exact solution invariant under rotation in y–z, dilation in space and u coordinates and the symmetry X_6 which seems to mimic Lorentzian spin in x.

3 Wave Equation in Spherically Symmetric Non-Static Flat Friedman Spacetime Background

In this section we discuss solution of the wave equation in a non-static spherically symmetric flat Friedman metric geometry [8]. The main purpose of our choosing this particular metric is that it is one of the three Friedman Universe models which do not have time translations invariance there. It may therefore be of interest to see how the nonlinearity shows up in the wave equation. To solve the wave equation in this metric we use Cartesian coordinates in which it becomes [5, 6],

$$ds^2 = dt^2 - t^{4/3}(dx^2 + dy^2 + dz^2). \tag{3.1}$$

Using (3.1) the wave equation (2.1) takes the form,

$$t^{4/3}u_{tt} + 2t^{1/3}u_t - (u_{xx} + u_{yy} + u_{zz}) = 0. \tag{3.2}$$

Equation (3.2) is a highly nonlinear partial differential equation which can be solved by following the procedure adopted in the previous example. We write the symmetry generator associated with this equation in the basis $\{t, x, y, z\}$ as,

$$V = p\frac{\partial}{\partial x} + q\frac{\partial}{\partial y} + r\frac{\partial}{\partial z} + n\frac{\partial}{\partial t} + f\frac{\partial}{\partial u} \tag{3.3}$$

We prolong this generator to include second derivatives and then use symmetry criterion (refer to (2.7)) to get, $V^1[(t^{4/3}u_{tt} + 2t^{1/3}u_t - (u_{xx} + u_{yy} + u_{zz})]|_{eq.(3.2)} = 0$. As before it is an algebraic equation in coefficients of derivatives of u of various orders. To deal with this equation we start by separating terms there by quadratic and cubic terms in the second derivatives of u. Integrating the resulting expressions it is immediately found that p, q, r are n are independent of u whilst f is linear in u. Proceeding with further separations we obtain the over determined system,

$$\begin{aligned}
&nt^{1/3} - t^{4/3}n_t - t^{7/3}f_{tu} + \frac{1}{2}t^{7/3}n_{tt} - \frac{1}{2}tn_{xx} - \frac{1}{2}tn_y - \frac{1}{2}tn_{zz} = 0\\
&\frac{1}{3}\frac{n}{t} - n_t + 2p_x + tf_{tu} - \frac{1}{2}tn_{tt} + \frac{1}{2t^{1/3}}(n_{xx} + n_{yy} + n_{zz}) = 0\\
&\frac{1}{3}\frac{n}{t} - n_t + 2q_y + tf_{tu} - \frac{1}{2}tn_{tt} + \frac{1}{2t^{1/3}}(n_{xx} + n_{yy} + n_{zz}) = 0\\
&\frac{1}{3}\frac{n}{t} - n_t + 2r_z + tf_{tu} - \frac{1}{2}tn_{tt} + \frac{1}{2t^{1/3}}(n_{xx} + n_{yy} + n_{zz}) = 0\\
&2n_y - 2t^{4/3}q_t = 0\\
&2n_x - 2t^{4/3}p_t = 0\\
&2n_z - 2t^{4/3}r_t = 0\\
&2p_z + 2r_x = 0\\
&2p_y + 2q_x = 0\\
&2q_z + 2r_y = 0\\
&-2t^{1/3}p_t - 2f_{xu} - t^{4/3}p_{tt} + p_{xx} + p_{yy} + p_{zz} = 0\\
&-2t^{1/3}q_t - 2f_{yu} - t^{4/3}q_{tt} + q_{xx} + q_{yy} + q_{zz} = 0\\
&-2t^{1/3}r_t - 2f_{zu} - t^{4/3}r_{tt} + r_{xx} + r_{yy} + r_{zz} = 0\\
&t^{4/3}f_{tt} + 2t^{1/3}f_t - (f_{xx} + f_{yy} + f_{zz}) = 0
\end{aligned} \tag{3.4}$$

As before one can solve the above equation either by hand or by crack software [11]. Using crack one immediately finds that the above system admits 10 dimensional algebra whose generators are given by,

$$
\begin{aligned}
&X_0=\frac{\partial}{\partial x},\qquad X_1=\frac{\partial}{\partial y},\qquad X_2=\frac{\partial}{\partial z},\\
&X_3=x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x},\qquad X_4=x\frac{\partial}{\partial z}-z\frac{\partial}{\partial x},\qquad X_5=y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y},\\
&X_6=-2xy\frac{\partial}{\partial x}+(-9t^{2/3}+x^2-y^2-z^2)\frac{\partial}{\partial y}-2zy\frac{\partial}{\partial z}-6ty\frac{\partial}{\partial t}+6yu\frac{\partial}{\partial u},\\
&X_7=(-9t^{2/3}-x^2+y^2+z^2)\frac{\partial}{\partial x}-2xy\frac{\partial}{\partial y}-2zx\frac{\partial}{\partial z}-6tx\frac{\partial}{\partial t}+6xu\frac{\partial}{\partial u},\\
&X_8=-2xz\frac{\partial}{\partial x}-2yz\frac{\partial}{\partial y}+(-9t^{2/3}+x^2+y^2-z^2)\frac{\partial}{\partial z}-6tz\frac{\partial}{\partial t}+6zu\frac{\partial}{\partial u},\\
&X_9=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}+2t\frac{\partial}{\partial t},
\end{aligned}
\tag{3.5}
$$

excluding the infinite one, $\phi(t,x,y,z)\partial_u$ where ϕ satisfies the wave equation (3.2). We now present reduction of the wave equation to an ordinary differential equation via the three-dimensional sub-algebra representing rotation in xy, dilation and translation in z, viz., $X_3 = x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}$, $X_9=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}+2t\frac{\partial}{\partial t}$ and $X_2=\frac{\partial}{\partial z}$. It is easy to check that each of the three symmetry generators commute with each other, namely their commutation relations are zero. Writing characteristic equation for the generator X_3 and solving it (cf. Sect. 2) yields invariants given by $\alpha=x^2+y^2$, t, z and u. Using these into the wave equation (3.2), with simultaneously applying translation in z, becomes

$$
t^{4/3}u_{tt}+2t^{1/3}u_t-(4u_{\alpha\alpha}+4u_\alpha)=0. \tag{3.6}
$$

Equation (3.6) inherits the generator X_9 which in the new coordinates becomes $2\alpha\frac{\partial}{\partial\alpha}+3t\frac{\partial}{\partial t}$ with invariants $\gamma=\frac{\alpha}{t^{2/3}}$ and u. Lastly, these invariants are applied to (3.6) and immediately reduce it to a nonlinear second order ordinary differential equation

$$
-\left(\frac{2}{9}\gamma+4\right)u_\gamma+4\left(\frac{1}{9}\gamma^2-\gamma\right)u_{\gamma\gamma}=0. \tag{3.7}
$$

A solution of this equation is given by

$$
u=c_1+\left\{54\arctan\left(\frac{\sqrt{\gamma-9}}{3}\right)+2\gamma\frac{\sqrt{\gamma-9}}{3}-24\sqrt{\gamma-9}\right\}c_2, \tag{3.8}
$$

where $\gamma=\alpha\, t^{\frac{-2}{3}}$ with $\alpha=x^2+y^2$.

4 Conservation Laws

In this section we give a brief procedure to use symmetry generators for variational equations and conservation laws via Noether's theorem. To illustrate this procedure we construct one conservation law in each of the spacetime geometries considered. A current $\Phi=(\Phi^1,\dots,\Phi^n)$ is conserved if it satisfies

$$
D_i\Phi^i=0 \tag{4.1}
$$

along the equation in question [12]. The Euler-Lagrange equations, if they exist, associated with the equation are the system $\delta L/\delta u^\alpha = 0$, $\alpha = 1, \dots, m$, where $\delta/\delta u^\alpha$ is the Euler-Lagrange operator given by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1}(-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial}{\partial u^\alpha_{i_1\cdots i_s}}, \quad \alpha = 1, \dots, m. \tag{4.2}$$

L is referred to as a Lagrangian and a Noether symmetry operator X of L arises from a study of the invariance properties of the associated functional

$$\mathcal{L} = \int_\Omega L(x, u, u_{(1)}, \dots, u_{(r)})\, dx \tag{4.3}$$

defined over Ω. If we include point dependent gauge terms $f_1, \dots, f_n$, the Noether symmetries X are given by

$$XL + LD_i\xi^i = D_i f_i. \tag{4.4}$$

Corresponding to each symmetry generator X, a conserved flow is obtained via Noether's theorem. The Lagrangian for the spacetime geometry given in (2.3) is,

$$L = \frac{1}{2}\left[\frac{a^2}{x}u_t^2 - xu_x^2 - xu_y^2 - xu_z^2\right] \tag{4.5}$$

Substituting this Lagrangian into (4.4), yields a subalgebra of Noether symmetries of (2.10); the generators X_4, X_{13} and X_{14} are not variational via this Lagrangian and, hence, will not contribute a conservation law. In all we can construct twelve conservation laws via the Noether's theorem. In particular, we use one symmetry generator, $X_0 = \frac{\partial}{\partial t}$, giving time translational invariance to construct one such example. Conservation laws corresponding to other symmetries can be similarly constructed. The variational conservation laws with respect to $X_0 = \frac{\partial}{\partial t}$ are given by,

$$\begin{aligned}
\Phi^1 &= -\frac{1}{2}\left(\frac{a^2}{x}u_t^2 + xu_x^2 + xu_y^2 + xu_z^2\right),\\
\Phi^2 &= xu_x u_t,\\
\Phi^3 &= xu_y u_t,\\
\Phi^4 &= xu_z u_t.
\end{aligned} \tag{4.6}$$

Of the four conservation laws, the Φ^2, Φ^3 and Φ^4 correspond to three components, T_{xt}, T_{yt} and T_{zt}, of the energy momentum tensor, while Φ^1 represents T_{tt} component. In particular, if we restrict u_x, u_y and u_z to zero, the Φ^1 can be seen as some analogue of energy of the spacetime. For the second spacetime metric (3.2) the Lagrangian is

$$L = \frac{1}{2}[t^2u_t^2 - t^{2/3}(u_x^2 + u_y^2 + u_z^2)]. \tag{4.7}$$

Due to particular nature of the geometry, the Lie symmetries of the wave equation compared with (3.5) are reduced, e.g, the time translational invariance does not exist any more. To construct one variational conservation law we choose translational symmetry $X_0 = \partial_x$.

This symmetry corresponds to linear momentum conservation and the four variational conservation laws associated with it are,

$$\begin{aligned}
\Phi^1 &= -t^2 u_x u_t,\\
\Phi^2 &= \frac{1}{2} t^2 u_t^2 + t^{2/3}(u_x^2 - u_y^2 - u_z^2),\\
\Phi^3 &= t^{2/3} u_x u_y,\\
\Phi^4 &= t^{2/3} u_x u_z.
\end{aligned} \tag{4.8}$$

In (2.3) the Φ^1 mimics conserved quantity representing momentum given by T_{xt} component of the energy momentum tensor, while Φ^2, Φ^3 and Φ^4 mimic three stresses in x-, y- and z- directions respectively. In particular, if we choose a frame in which the three velocities become zero the Φ^2 defines some analogue of energy of the spacetime. Conservation laws corresponding to remaining symmetries can be similarly be constructed.

5 Discussion and Conclusion

We have considered the classical wave equation in some Lorentzian spacetime backgrounds with a point in mind that the wave equation there may naturally inherit nonlinearity from geometry. In this connection we have considered two spacetime metrics which respectively represent a plane symmetric static metric [7] and flat Friedmann metric of signature -2. For both cases we have given solutions each to show how wave equation there can be either solved or reduced to ordinary differential equations by using the method of invariants. yielding additional conservation laws that were not given previously. In his book [4] Ibragimov suggests that in three flat space dimensions the linear wave equation admits 16-dimensional Lie algebra of point symmetries excluding the 'infinite symmetry'. In this study we show that the wave equations admits fewer symmetries when it is solved on general Lorentzian manifolds. In particular we have shown that the wave equations in plane symmetric static spacetime admits 15 Lie point symmetries which are one less than 16 Lie point symmetries of the wave equation in 3 Cartesian space dimensions suggested in [4]. It is presumably a shift away effect from 'flatness' from Minkowski manifold to other manifolds leading to reduction in symmetry as well as solutions of the wave equation. This shift away from flatness of Minkowski manifold is more clear in flat Friedmann metric case where only 10 Lie point symmetries of the wave equation are recovered. There the shift away from flatness is by 6 symmetry generators [4]. In fact, the alternatives to the Minkowski case do not lend themselves to the variational case as conveniently as does the Minkowski case. Also, it should be noted that these alternatives to the wave equation on the Minkowski manifold are not achievable via a simple point transformation of variables on the Minkowski version. It is hoped that solving fully nonlinear wave equation in curved spacetime background using Lie symmetry methods may provide some insight in geometry or relativity for different manifolds.

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